# Air entrainment in transient flows in closed water pipes: a two-layer approach.

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**Abstract**

In this paper, we first construct a model for transient free surface flows that takes into account the air entrainment by a sytem of 4 partial differential equations. We derive it by taking averaged values of gas and fluid velocities on the cross surface flow in the Euler equations (incompressible for the fluid and compressible for the gas). Then, we propose a mathematical kinetic interpretation of this system to finally construct a well-balanced kinetic scheme having the properties of conserving the still water steady state and possesing an energy. Finally, numerical tests on closed uniforms water pipes are performed and discussed.

**Keywords**: Two-layer model, pressurised air flows, free surface flows, well-balanced kinetic scheme, non conservative product.

## 1 Introduction

The hydraulic transients of flow in pipes have been investigated by Hamam and McCorquodale [12], Song et al. [19, 20, 21], Wylie and Streeter [25] and others. However, the air flow in the pipeline and associated air pressure surge was not studied extensively.
Two component gas-liquid mixture flows occur in piping systems in several industrial areas such as nuclear power plants, petroleum industries, geothermal power plants, pumping stations and sewage pipelines. Gas may be entrained in other liquid-carrying pipelines due to cavitation, [24] or gas release from solution due to a drop or increase in pressure.
Unlike in a pure liquid in which the pressure wave velocity is constant, the wave velocity in a gas-liquid mixture varies with the pressure. Thus the main coefficients in the conservation equation of the momentum are pressure dependent and consequently the analysis of transients in the two-component flows is more complex.
The most widely used analytical models for two-component fluid transients are the homogeneous model [9], the drift flux model [10, 13, 14], and the separated flow model, [22, 23].
In the homogeneous model, the two phases are treated as a single pseudo-fluid with averaged properties [9, 25]: there is no relative motion or slip between the two phases. The governing equations are the same as the one for a single phase flow. However the inertial and gravitationnal effects can play a more important role than the relative velocity between the air phase and the liquid one: the interaction between the two phases has to be taken into account.

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In the drift-flux model, [10, 13, 14], the velocity fields are expressed in terms of the mixture center-of-mass velocity and the drift velocity of the vapor phase, which is the vapor velocity with respect to the volume center of the mixture. The effects of thermal non-equilibrium are accommodated in the drift-flux model by a constitutive equation for phase change that specifies the rate of mass transfer per unit volume. Since the rates of mass and momentum transfer at the interfaces depend on the structure of two-phase flows, these constitutive equations for the drift velocity and the vapor generation are functions of flow regimes.
Two-phase flows always involve some relative motion of one phase with respect to the other; therefore, a two-phase flow problem should be formulated in terms of two velocity fields. A general transient two-phase flow problem can be formulated by using a two-fluid model. The separated flow model considers the two phases individually, interacting with each other. Generally, this model will be written as a six equations partial differential system that represent the balance of mass, momentum and energy [22, 23].
In the present paper, we consider the air entrainment appearing in the transient flow in closed pipes not completely filled: the liquid flow (as well as the air flow) is free surface. We thus derived the present model in a very closed manner as the derivation of Saint-Venant equations for free surface open channel flow: we consider that the liquid is incompressible and its pressure is only due to the gravitationnal effect. It is the well known hydrostatic pressure law. The air phase is supposed to be compressible and isothermal: thus we can use an equation of state of the form $p = k\rho^\gamma$ where $\gamma$ represent the adiabatic exponent. To connect the two phases, we write the continuity of the normal stress tensor at the free surface separating the two phases: the hydrostatic pressure law for the fluid is thus coupled to the pressure law governing the air flow at the free surface. Then by taking averaged values over cross-sections of the main flow axis, we obtain a four equations partial differential system: we called it the two-layer water/air model. We state in theorem 2.1 its main properties: the existence of a total energy despite the fact that the obtained system is not strictly hyperbolic (except for the particular case where the averaged speeds of the liquid and the gas are the same). The derivation of the proposed model and the investigation of the mathematical properties are presented in section 2.
As we will see, the main difficulty in the construction of a numerical scheme to solve the obtained model comes from the fact that the proposed model is not hyperbolic: the eigenvalues of the jacobian of the flux may be complex. Moreover we cannot compute exactly these eigenvalues since we do not suppose that the velocity of each phase are close. This is the main reason why we chose to interpret this model as a kinetic model: we present the mathematical kinetic formulation of the four equations partial differential equations in the section 3. This mathematical formulation permits us to construct a well-balanced kinetic scheme presented in the section 4
Finally, we propose in section 5 some numerical simulation to focus on the role of the air entrainment on the free surface fluid flow.

# 2 A two-layer water/air model

We will consider throughout this paper a transient flow in a closed pipe that is not completely filled of water nor of air. The flow is composed by an air and water layer separated by a moving interface. We assume that the two layers are immiscibles. We will consider that the fluid is perfect and incompressible whereas the air is a perfect compressible gas: we neglect here the air-water interactions as condensation/evaporation and we focus only on the compressibility effects of the air on the water layer. The air layer is supposed to be isothermal and isentropic. The derivation of the two-layer model is then performed with the use of the 3D Euler (incompressible for the fluid, compressible for the gas) equations.

**Remark 2.1** *Throughout the paper, we use the following notations: we will denote by the index $w$ the water layer and by the index $a$ the air layer. We also use to the index $\alpha$ to write indifferently the liquid or gas layer.*

The pipe is assumed to have a symetry axis $\mathcal{C}$ which is represented by a straight line with, for the sake of simplicity, a constant angle $\theta$ with the horizontal direction. $(O, \mathbf{i}, , \mathbf{j}, \mathbf{k})$ is a reference frame atached to this axis with $\mathbf{k}$ orthogonal to $\mathbf{i}$ in the vertical plane containg $\mathcal{C}$, see Fig. 1. We denote by $\Omega_{t,a}$ and $\Omega_{t,w}$ the gas

and fluid domain occupied at time $t$. Then, at each point $\omega(x)$, we define the water section $\Omega_w(t,x)$ by the set:

$$\Omega_w(t,x) = \left\{(y,z) \in \mathbb{R}^2; z \in [-R(x), -R(x) + H_w(t,x)],\ y \in [\beta_l(x,z), \beta_r(x,z)]\right\}$$

and the air one by the set:

$$\Omega_a(t,x) = \left\{(y,z) \in \mathbb{R}^2; z \in [-R(x) + H_w(t,x), R(x)],\ y \in [\beta_l(x,z), \beta_r(x,z)]\right\}$$

where $R(x)$ denotes the radius, $H_w(t,x)$ the water height at section $\Omega_w(t,x)$, $H_a(t,x)$ the air height at section $\Omega_a(t,x)$. $\beta_l(x,z)$, $\beta_r(x,z)$ are respectively the left and right $y$ values of the boundary points of the domain at altitude $-R(x) < z < R(x)$ (see Fig. 1 and Fig. 2)

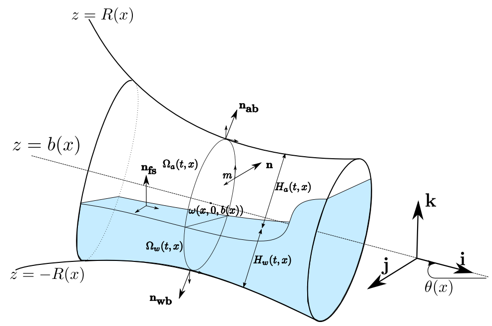



Figure 1: Geometric characteristics of the domain

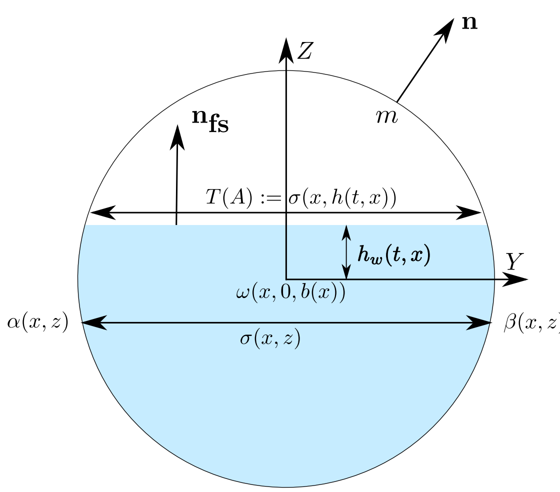



Figure 2: Cross-section of the domain

We have then the first natural coupling:

$$H_w(t,x) + H_a(t,x) = 2R(x)\,. \tag{1}$$

In what follows, we will denote $h_w(t,x) = -R(x) + H_w(t,x)$ the algebraic water height.

## 2.1 The fluid free surface model

The shallow water like equations for free surface flows are obtained from the incompressible Euler equations (2) which writes in cartesian coordinates $(x, y, z)$:

$$\begin{array}{rcll} \mathrm{div}(\mathbf{U_w}) & = & 0, & \text{on } \mathbb{R} \times \Omega_{t,w} \\ \partial_t(\mathbf{U_w}) + \mathrm{div}(\mathbf{U_w} \otimes \mathbf{U_w}) + \nabla P_w & = & \mathbf{F}, & \text{on } \mathbb{R} \times \Omega_{t,w} \end{array} \quad (2)$$

where $\mathbf{U_w}(t, x, y, z) = (U_w, V_w, W_w)$, $P_w(t, x, y, z)$, $\mathbf{F}$ denote respectively the fluid velocity, the scalar pressure and the exterior strength which represents the gravity.
A kinematic law at the free surface and a no-leak condition $\mathbf{U_w}.\mathbf{n_{wb}} = 0$ on the wetted boundary ($\mathbf{n_{wb}}$ beeing the outward unit normal vector FIG. 2) complete the system (see the works of Bourdarias *et al.* for a complete derivation [5, 6, 7]).
To model the effects of the air on the water, we write the continuity of the normal stress at the free boundary layer separated the two phases. Thus the water pressure $P_w$ may be writen as

$$g(h_w - z)\cos\theta + P_a(\overline{\rho})/\rho_0$$

where the term $g(h_w - z)\cos\theta$ represents the hydrostatic pressure whereas $P_a(\overline{\rho})$ represents the air pressure, $\rho(t, x, y, z) = \rho_0$ represents the density of water at atmospheric pressure and $\overline{\rho}$ is the mean density of the air over the section $\Omega_a(t, x)$. This is the second natural coupling between the two phases.
Let us now introduce the conservative variables $A(t, x)$ and $Q(t, x) = A(t, x)u(t, x)$ representing respectively the wetted area and the discharge defined as:

$$A(t, x) = \int_{\Omega_w} dydz, \quad Q(t, x) = A(t, x)u(t, x)$$

where $u$ is the mean value of the speed

$$u(t, x) = \frac{1}{A(t, x)} \int_{\Omega_w} U_w(t, x, y, z)\, dydz.$$

We take averaged values along the cross-section $\Omega_w(t, x)$ in the incompressible Euler equations, and we approximate the averaged value of a product by the product of the averaged values ( see for instance [1, 5, 6, 7]). This leads to the free surface model:

$$\left\{ \begin{array}{rcl} \partial_t A + \partial_x Q & = & 0 \\ \partial_t Q + \partial_x \left( \dfrac{Q^2}{A} + AP_a(\overline{\rho})/\rho_0 + gI_1(x, A)\cos\theta \right) & = & -gA\partial_x Z \\ & & +gI_2(x, A)\cos\theta \\ & & +P_a(\overline{\rho})/\rho_0\, \partial_x A \end{array} \right. \quad (3)$$

where $g$ is the gravity constant an $Z(x)$ is the elevation of the point $\omega(x)$. The terms $I_1(x, A)$ and $I_2(x, A)$ are defined by:

$$I_1(x, A) = \int_{-R}^{h_w} (h_w - Z)\sigma(x, z)\, dz$$

and

$$I_2(x, A) = \int_{-R}^{h_w} (h_w - Z)\partial_x \sigma(x, z)\, dz\,.$$

They represent respectively the term of hydrostatic pressure and the pressure source term. In theses formulas $\sigma(x, z)$ is the width of the cross-section at position $x$ and at height $z$ (see FIG. 2).

## 2.2 The air layer model

The derivation of the air layer model is based on the Euler compressible equations (4) which writes in cartesian coordinates $(x, y, z)$:

$$\begin{array}{rcll} \partial_t \rho_a + \mathrm{div}(\rho_a \mathbf{U_a}) & = & 0, & \text{on } \mathbb{R} \times \Omega_{t,a} \\ \partial_t(\rho_a \mathbf{U_a}) + \mathrm{div}(\rho_a \mathbf{U_a} \otimes \mathbf{U_a}) + \nabla P_a & = & 0, & \text{on } \mathbb{R} \times \Omega_{t,a} \end{array} \tag{4}$$

where $\mathbf{U_a}(t, x, y, z) = (U_a, V_a, W_a)$ and $\rho_a(t, x, y, z)$ denotes the velocity and the density of the air whereas $P_a(t, x, y, z)$ is the scalar pressure. Let us mention here that we neglected the gravitationnal effect on the air layer. We neglect the second and third equation in the momentum equations to assume that the mean air flow follows the $x$-axis.

**Remark 2.2** *We can also obtain it from a classical rescaling $\epsilon = 2R/L$ where $R$ is the radius of the pipe section, $L$ its length and we get in particular the equations $\partial_y p_a = \partial_z p_a = 0$. Thus in the following development, we may assume that the density of air depends only on $t$ and $x$, see [5] for a formal asymptotic derivation of the model for mixed flows in closed pipes.*

We use the standard notation for the mean value of the density of air $\overline{\rho}$ over the air section $\Omega_a$. We assume that the air layer is isentropic, isothermal, immiscible and follows a perfect gaz law. Thus we have the following equation of state:

$$P_a(\rho) = k\,\rho^\gamma \text{ with } k = \frac{p_a}{\rho_a^\gamma} \tag{5}$$

for some reference pressure $p_a$ and density $\rho_a$. The adiabatic index $\gamma$ is set to 7/5.
With the definition of the free surface pressure at Subsection 2.1 and the air pressure definition above and assuming that the velocity is normal at the free surface of each layer, the two-layer model follows the continuity property of the normal stress for two immiscible perfect fluids at the interface.
Let us now introduce the air area $\mathcal{A}$ by:

$$\mathcal{A} = \int_{\Omega_a} dydz$$

and the averaged air velocity $v$ (we recall that $u$ is the averaged velocity of the fluid layer) by:

$$v(t, x) = \frac{1}{\mathcal{A}(t, x)} \int_{\Omega_a} U_a(t, x, y, z)\, dydz \quad .$$

The conservative variables are $M = \overline{\rho}/\rho_0 \mathcal{A}$ and $D = Mv$ and represent the rescaled air mass and discharge. Let us remark that we use rescaled mass and discharge instead of the real mass and discharge to be consistent with the free surface model (3). As done before to obtain the water layer model, we take averaged value in the Euler equations (4) over sections $\Omega_a$, and perform the same approximations on averaged values of a product to obtain the following model:

$$\left\{ \begin{array}{rcl} \partial_t M + \partial_x D & = & \displaystyle\int_{\partial\Omega_a} \rho/\rho_0\ (\partial_t \mathbf{m} + v\partial_x \mathbf{m} - \mathbf{v})\,.\mathbf{n}\, ds \\ \partial_t D + \partial_x \left( \dfrac{D^2}{M} + P_a(\overline{\rho})/\rho_0\, \mathcal{A} \right) & = & P_a(\overline{\rho})/\rho_0\, \partial_x(\mathcal{A}) \\ & & + \displaystyle\int_{\partial\Omega_a} \rho/\rho_0\, v\, (\partial_t \mathbf{m} + v\partial_x \mathbf{m} - \mathbf{v})\,.\mathbf{n}\, ds \end{array} \right. \tag{6}$$

where $\mathbf{v}$ is the velocity in the $(\mathbf{j}, \mathbf{k})$-plane. The boundary $\partial\Omega_a$ is divided into the free surface boundary $\Gamma_{fs}$ and the border of the pipe in contact with air $\Gamma_c$. For $m \in \Gamma_c$, $\mathbf{n} = \dfrac{\mathbf{m}}{|\mathbf{m}|}$ is the outward unit vector at the point $m$ in the $\Omega$-plane and $\mathbf{m}$ stands for the vector $\omega m$ while $\mathbf{n}$ denotes $-\mathbf{n}_{fs}$ on $\Gamma_{fs}$. As the pipe is infinitely rigid, the non-penetration condition holds on $\Gamma_c$, namely: $\mathbf{U_a}.\mathbf{n}_{ab} = 0$ (see Fig. 1).

On the free surface, as the boundary kinematic condition used for the water layer at the free surface holds, the integral appearing in System (6) vanishes. Using the equation of state (5), and using the air sound speed defined by

$$c_a^2 = \frac{\partial p}{\partial \rho} = k\gamma \left(\frac{\rho_0 M}{\mathcal{A}}\right)^{\gamma-1} \quad , \tag{7}$$

we get finaly the air-layer model:

$$\left\{\begin{array}{rcl} \partial_t M + \partial_x D &=& 0 \\ \partial_t D + \partial_x \left(\dfrac{D^2}{M} + \dfrac{M}{\gamma} c_a^2\right) &=& \dfrac{M}{\gamma} c_a^2\, \partial_x(\mathcal{A}) \end{array}\right. . \tag{8}$$

### 2.3 The two-layer model

The two-layer model is then simply obtain by the apposition of the models (3) and (8) and using the coupling (1) which also writes $\mathcal{A} + A = S$ where $S = S(x)$ denotes the pipe section:

$$\left\{\begin{array}{rcl} \partial_t M + \partial_x D &=& 0 \\ \partial_t D + \partial_x \left(\dfrac{D^2}{M} + \dfrac{M}{\gamma} c_a^2\right) &=& \dfrac{M}{\gamma} c_a^2\, \partial_x(S - A) \\ \\ \partial_t A + \partial_x Q &=& 0 \\ \partial_t Q + \partial_x \left(\dfrac{Q^2}{A} + gI_1(x, A)\cos\theta + A\dfrac{c_a^2\, M}{\gamma\,(S - A)}\right) &=& -gA\partial_x Z + gI_2(x, A)\cos\theta \\ && + \dfrac{c_a^2\, M}{\gamma\,(S - A)}\, \partial_x A \end{array}\right. . \tag{9}$$

The two-layer model has the inconvenient to be non hyperbolic since we may have complex eigenvalues as we will see. Moreover, we have no way to give it explicitly. Indeed, setting the conservative variable $\mathbf{W} = (M, D, A, Q)^t$, let us consider System (9) with $\theta = 0$ and $S(x) = S$ (for the sake of simplicity) written under the non conservative form:

$$\partial_t \mathbf{W} + \mathcal{D}(\mathbf{W}) \partial_X \mathbf{W} = TS(\mathbf{W})\,. \tag{10}$$

with the convection matrix:

$$\mathcal{D} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ c_a^2 - v^2 & 2v & \dfrac{M}{S - A} c_a^2 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{A}{(S - A)} c_a^2 & 0 & c_w^2 + \dfrac{AM}{(S - A)^2} c_a^2 - u^2 & 2u \end{pmatrix}$$

where $c_a$ being the air sound speed defined by (7) and $c_w = \sqrt{g \dfrac{A}{T(A)} \cos\theta}$ the water sound speed. $T(A)$ is the width of the free surface at height $h_w(A)$.

Since the angle $\theta$ is set to 0 and $S$ is constant, the source term $TS(\mathbf{W})$ vanishes. The eigenvalues $\lambda$ of the convection matrix must satisfy the following polynomial equation:

$$\left(\lambda^2 - 2v\lambda + \left(c_a^2 - v^2\right)\right) \left(\lambda^2 - 2u\lambda - \left(c_w^2 + \frac{AM}{(S - A)^2} c_a^2 - u^2\right)\right) = \frac{AM}{(S - A)^2} c_a^4\,.$$

**Remark 2.3** *We can remark that when the speed of each layer are equal or even close, we can write, explicitly in the case $u = v$ or an asymptotic development in the case where $u \approx v$, the four distinct eigenvalues:*

$$u \pm \frac{1}{2}\sqrt{2\left(c_a^2\left(1+\frac{AM}{(S-A)^2}\right)+c_w^2\right) \pm \sqrt{\left(c_a^2\left(1-\frac{AM}{(S-A)^2}\right)+c_w^2\right)^2 + 4c_a^4\frac{AM}{(S-A)^2}}}$$

*Two of them are real and two are complex. This makes the system non-hyperbolic. Thus the classical Roe like Finite Volume method is not appropriate since it is necessary to solve a Riemann problem at the interfaces of the cells which asks to compute the eigenvalues.*
*Moreover, if we assume that the air density is small enough (e.g. $c_a \approx 0$), the eigenvalues are simply $v, v, u \pm c_w$. However, in the case of a high speed hydraulic jump, the air speed can reach severe values and may be very different of the speed of the water. Also the air layer may be strongly compressed and the eigenvalues cannot be computed.*
*Nonetheless, the problem of complex eigenvalues can be solved easily in a kinetic framework (see sections 3 and 4).*

Although System (9) is not hyperbolic when $u$ is different from $v$, we may construct a non increasing total energy, which permits us to think that there exist entropic solutions of System (9) (see also [3]). The results are summarized in the following theorem:

**Theorem 2.1**

1. *For smooth solutions, the velocity $u$ and $v$ satisfies*

$$\partial_t v + \partial_x\left(\frac{v^2}{2} + \frac{c_a^2}{\gamma-1}\right) = 0 \tag{11}$$

$$\partial_t u + \partial_x\left(\frac{u^2}{2} + g h_w(A)\cos\theta + gZ + \frac{c_a^2 M}{\gamma(S-A)}\right) = 0 \quad . \tag{12}$$

2. *System (9) admits a mathematical total energy*

$$E = E_a + E_w$$

*with*

$$E_a = \frac{Mv^2}{2} + \frac{c_a^2 M}{\gamma(\gamma-1)}$$

*and*

$$E_w = \frac{Au^2}{2} + gA(h_w - I_1(x,A)/A)\cos\theta + gAZ)$$

*which satisfies the entropy equality*

$$\partial_t E + \partial_x H = 0$$

*where the flux $H$ is the total entropy flux defined by*

$$H = H_a + H_w$$

*where the air entropy flux reads:*

$$H_a = \left(E_a + \frac{c_a^2 M}{\gamma}\right)v$$

*and the water entropy flux reads:*

$$H_w = \left(E_w + gI_1(x,A)\cos\theta + A\frac{c_a^2 M}{(S-A)}\right)u \quad .$$

*3. The energy $E_a$ and $E_w$ satisfies the following entropy flux equalities:*

$$\partial_t E_a + \partial_x H_a = \frac{c_a^2\, M}{\gamma(S-A)}\,\partial_t A \tag{13}$$

*and*

$$\partial_t E_w + \partial_x H_w = -\frac{c_a^2\, M}{\gamma(S-A)}\,\partial_t A \quad . \tag{14}$$

■

**Proof of theorem 2.1**:
The proof of these results relies only on algebraic computations and combinations of the equations forming System (9).

**Remark 2.4** *Let us remark that Equations (13)-(14) can be interpreted in such a way: the system is physically closed e.g. the energy dissipated (or gained) by the water layer is forwarded to (or lost by) the air layer. This property ensures the total energy equality $\partial_t E + \partial_x H = 0$.*

# 3 The mathematical kinetic formulation

As pointed out before (see Subsection 2.3), Roe like Finite Volume method is not appropriate since, in our case, the eigenvalues cannot be computed explicitly and approximations cannot be used unlike the multilayer Saint-Venant system solved by a Roe like Finite Volume method in [2].
A kinetic approach is proposed to overcome this difficulty. We recall the kinetic formulation (see e.g. [17]) and we apply it to the two-layer model (9). Let us consider a smooth real function $\chi$ which has the following properties:

$$\chi(\omega) = \chi(-\omega) \geq 0\ ,\ \int_{\mathbb{R}} \chi(\omega)d\omega = 1, \int_{\mathbb{R}} \omega^2 \chi(\omega)d\omega = 1 \quad . \tag{15}$$

We define the Gibbs equilibrium as

$$\mathcal{M}_\alpha(t,x,\xi) = \frac{A_\alpha}{b_\alpha}\chi\left(\frac{\xi - u_\alpha}{b_\alpha}\right)$$

which represents the density of particles at time $t$, position $x$ with the kinetic speed $\xi$ with

$$b_\alpha^2 = \left\{ \begin{array}{ll} \dfrac{c_a^2}{M} & \text{if} \quad \alpha = a \\[2ex] g\dfrac{I_1(x,A)}{A}\cos\theta + \dfrac{c_a^2\, M}{\gamma\,(S-A)} & \text{if} \quad \alpha = w \end{array} \right. .$$

We denote:

$$A_\alpha = \left\{ \begin{array}{lll} A & \text{if} & \alpha = w \\ M & \text{if} & \alpha = a \end{array} \right. ,\quad Q_\alpha = \left\{ \begin{array}{lll} Q & \text{if} & \alpha = w \\ D & \text{if} & \alpha = a \end{array} \right. \text{and } u_\alpha = \left\{ \begin{array}{lll} u & \text{if} & \alpha = w \\ v & \text{if} & \alpha = a \end{array} \right. .$$

Thus we may state the following kinetic interpretation of System (9):

**Theorem 3.1** *The couple of functions $(A_\alpha, Q_\alpha)$ is a strong solution of System (9) if and only if $\mathcal{M}_\alpha$ satisfies the two kinetic transport equations (one per phase):*

$$\partial_t \mathcal{M}_\alpha + \xi \partial_X \mathcal{M}_\alpha + \phi_\alpha \partial_\xi \mathcal{M}_\alpha = K_\alpha(t,x,\xi) \quad \textit{for } \alpha = a \textit{ and } \alpha = w \tag{16}$$

*for some collision term $K_\alpha(t,x,\xi)$ which satisfies for a.e. $(t,x)$*

$$\int_{\mathbb{R}} K_\alpha\, d\xi = 0\ ,\ \int_{\mathbb{R}} \xi\, K_\alpha d\,\xi = 0 \quad . \tag{17}$$

*The function $\phi_\alpha$ is defined by:*

$$\phi_\alpha = \begin{cases} -\dfrac{c_a^2}{S-A}\partial_x(S-A) & \textit{if} \quad \alpha = A \\ g\partial_x Z - g\dfrac{I_2(x,A)}{A}\cos\theta - \dfrac{M}{S-A}c_a^2\,\partial_x \ln(A) & \textit{if} \quad \alpha = W \end{cases} \tag{18}$$

■

**Proof of theorem 3.1** The proof relies on a very obvious computation. Indeed, the equations forming the System (9) are obtained by taking the moments of the two kinetic Equations (16) with respect to $\xi$ against 1, $\xi$ and $\xi^2$: the right-hand side vanishes according to (17) and the left-hand sides coincides exactly thanks to (15). These are consequences of the following relations verified by the microscopic equilibrium:

$$A_\alpha = \int_{\mathbb{R}} \mathcal{M}_\alpha(t,x,\xi)\,d\xi$$

$$Q_\alpha = \int_{\mathbb{R}} \xi\mathcal{M}_\alpha(t,x,\xi)\,d\xi$$

$$\frac{Q_\alpha^2}{A_\alpha} + b_\alpha^2 A_\alpha = \int_{\mathbb{R}} \xi^2\mathcal{M}_\alpha(t,x,\xi)\,d\xi$$

■

**Remark 3.1** *The previous results (3.1) are obtained assuming a constant angle $\theta$. The generalization of these results with varying $\theta = \theta(x)$ can be found in [5] for the free surface layer.*

# 4 The kinetic scheme

This section is devoted to the construction of the numerical kinetic scheme and its properties. The numerical scheme is obtained by using the previous kinetic formulation (16). The term $\phi_\alpha$ is easily upwinded. Indeed, the conservative term $\partial_x Z$ is a given piecewise constant function and all n on conservative products of $\phi_\alpha$ are defined following, for instance [11, 15, 16]. For the sake of simplicity, we will only consider uniform pipes, that induces $S(x) = S$.
In the sequel, for the sake of simplicity, we consider infinite space domain. Let us consider the discretisation $(m_i)_{i\in\mathbb{Z}}$ of the spatial domain with

$$m_i = (x_{i-1/2}, x_{i+1/2}),\ h_i = x_{i+1/2} - x_{i-1/2},\ i \in \mathbb{Z}$$

which are respectively the cell and mesh size. Let $\Delta t^n = t_{n+1} - t_n$, $n \in \mathbb{N}$ be the timestep.
Let $W_{\alpha,i}^n = (A_{\alpha,i}^n, Q_{\alpha,i}^n)$, $u_{\alpha,i}^n = \dfrac{Q_{\alpha,i}^n}{A_{\alpha,i}^n}$ be respectively the approximation of the mean value of $(A_\alpha, Q_\alpha)$ and the velocity $u_\alpha$ on $m_i$ at time $t_n$. Let $\mathcal{M}_{\alpha,i}^n(\xi) = \dfrac{A_{\alpha,i}^n}{b_{\alpha,i}^n}\chi\left(\dfrac{\xi - u_{\alpha,i}^n}{b_{\alpha,i}^n}\right)$ be the approximation of the microscopic quantities and $\phi_{\alpha,i}^n \mathbb{1}_{m_i}(X)$ of $\phi_\alpha$. Then, integrating System (9) over $]x_{i-1/2}, x_{i+\frac{1}{2}}[\times[t_n, t_{n+1}[$, we can write a Finite Volume scheme as follows:

$$\mathbf{W}_{\alpha,i}^{n+1} = \mathbf{W}_{\alpha,i}^n - \frac{\Delta t^n}{h_i}\left(\mathbf{F}_{\alpha,i+1/2}^- - \mathbf{F}_{\alpha,i-1/2}^+\right). \tag{19}$$

where the numerical flux $\mathbf{F}_{\alpha,i\pm1/2}^\pm$ are defined from the microscopic fluxes $\mathcal{M}_{\alpha,i\pm1/2}^\pm$ with the relation:

$$\mathbf{F}_{\alpha,i\pm1/2}^\pm = \int_{\mathbb{R}} \begin{pmatrix} \xi \\ \xi^2 \end{pmatrix} \mathcal{M}_{\alpha,i\pm1/2}^\pm(\xi)\,d\xi\,.$$

The microscopic fluxes $\mathcal{M}^{\pm}_{\alpha,i\pm 1/2}$ are upwinded by the following formulas as done in [4, 8, 18]:

$$
\begin{aligned}
\mathcal{M}^{-}_{\alpha,i+1/2}(\xi) &= \overbrace{\mathbb{1}_{\{\xi>0\}}\mathcal{M}^{n}_{\alpha,i}(\xi)}^{\text{positive transmission}} + \overbrace{\mathbb{1}_{\left\{\xi<0,\xi^2-2g\Delta\phi_{\alpha,i+1/2}<0\right\}}\mathcal{M}^{n}_{\alpha,i}(-\xi)}^{\text{reflection}} \\
&+ \underbrace{\mathbb{1}_{\left\{\xi<0,\xi^2-2g\Delta\phi_{\alpha,i+1/2}>0\right\}}\mathcal{M}^{n}_{\alpha,i+1}\left(-\sqrt{\xi^2-2g\Delta\phi_{\alpha,i+1/2}}\right)}_{\text{negative transmission}} \\
\mathcal{M}^{+}_{\alpha,i+1/2}(\xi) &= \overbrace{\mathbb{1}_{\{\xi<0\}}\mathcal{M}^{n}_{\alpha,i+1}(\xi)}^{\text{negative transmission}} + \overbrace{\mathbb{1}_{\left\{\xi>0,\xi^2+2g\Delta\phi_{\alpha,i+1/2}<0\right\}}\mathcal{M}^{n}_{\alpha,i+1}(-\xi)}^{\text{reflection}} \\
&+ \underbrace{\mathbb{1}_{\left\{\xi>0,\xi^2+2g\Delta\phi_{\alpha,i+1/2}>0\right\}}\mathcal{M}^{n}_{\alpha,i}\left(\sqrt{\xi^2+2g\Delta\phi_{\alpha,i+1/2}}\right)}_{\text{positive transmission}}
\end{aligned}
\tag{20}
$$

with

$$
\Delta\phi_{\alpha,i+1/2} = \begin{cases} \dfrac{\widetilde{(c_a^2)}}{\gamma}\,(A_{i+1}-A_i) & \text{if} \quad \alpha = a \\[2ex] g\,(Z_{i+1}-Z_i) - \left(\widetilde{\dfrac{c_a^2 M}{S-A}}\right)\dfrac{1}{\gamma}\,\ln(A_{i+1}/A_i) & \text{if} \quad \alpha = w \end{cases} .
$$

To construct this upwinding fluxes, we used firstly the fact that every quantities involved are piecewise constant functions and may define non conservative products. Using the "straight lines" paths:

$$
\Phi(s, W_i, W_{i+1}) = sW_{i+1} + (1-s)W_i,\; s \in [0,1]
$$

(see e.g. [11, 15, 16] and Fig. 3) with $W_i$, $W_{i+1}$ the left and right state at the discontinuity $x_{i+1/2}$ permits us to approach any non conservative product $f(x,W)\partial_x W$ as $\widetilde{f}(W_{i+1}-W_i)$ with the notation

$$
\tilde{f} = \int_0^1 f(x_{i+1/2}, \Phi(s, W_i, W_{i+1}))\, ds \quad .
$$

$\Delta\phi_{\alpha,i\pm 1/2}$ denotes the upwinded source terms corresponding to (18) and $\xi^2 \pm 2\Delta\phi_{\alpha,i+1/2}$ is the jump condition for a particle with the kinetic speed $\xi$ which is necessary to:

- be reflected: this means that the particle has not enough kinetic energy $\xi^2/2$ to overpass the potential barrer which corresponds to potential energy $\Delta\phi_\alpha$,
- overpass the potential barer with a positive speed,
- overpass the potential barer with a negative speed,

as displayed on Fig. 3.

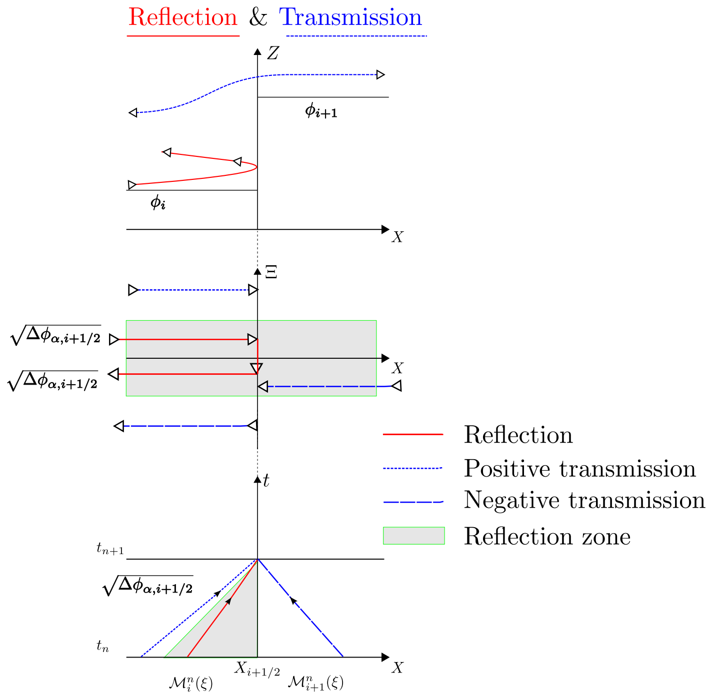



Figure 3: The potential barer: transmission and reflection of particle
Top: the physical configuration
Middle: the characteristic solution in $(X, \Xi)$-plane
Bottom: the characteristic solution in $(X, t)$-plane

We use as $\chi$ function the function defined by $\chi(w) = \dfrac{1}{2\sqrt{3}} \mathbb{1}_{[-\sqrt{3},\sqrt{3}]}(w)$ . Thus a stability CFL condition is necessary and under this condition, we can state the following results on the positivity of the air and water areas, the still steady states and the flooding and drying process:

**Theorem 4.1** *Assuming the CFL condition*

$$\Delta t^n < \min_{i\in\mathbb{Z}} \left( \frac{h_i}{\max_\alpha \left(|u^n_{\alpha,i}| + \sqrt{3} c^n_{\alpha,i}\right)} \right)$$

*holds, we have:*

1. *the numerical kinetic scheme keeps $A_\alpha$ positive.*
2. *The still air/water steady state is preserved:*

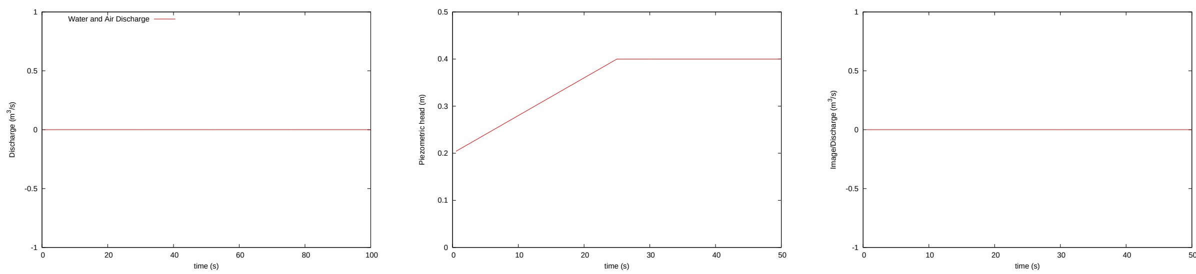
$$\begin{aligned} &u_i^n = 0\ ,\ v_i^n = 0 \\ &h^n_{w,i} \cos\theta_i + Z_i + \frac{M_i^n}{g\gamma(S_i - A_i^n)} {c^n_{a,i}}^2 = cst_1 \\ &\frac{{c^n_{a,i}}^2}{\gamma - 1} = cst_2 \end{aligned}$$

3. *Drying and flooding areas are also treated.*

# 5 Numerical tests

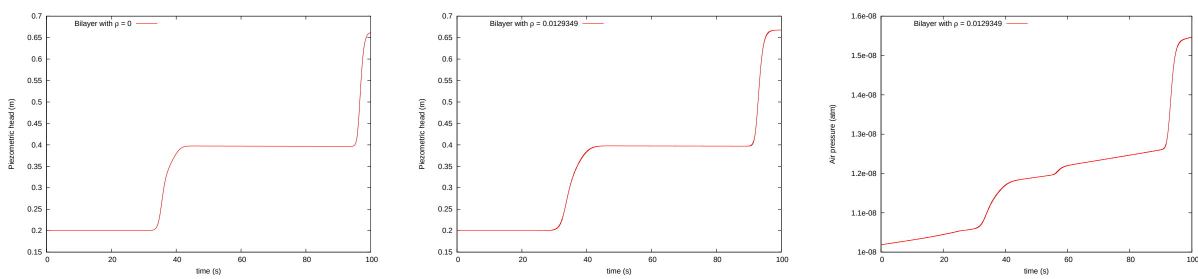
The numerical validation consist in showing the air entrainment effect in closed pipes with constant section and no slope. Since experimental data for such flows in any pipes are not available, we focus only the qualitative behavior of our model and our method for different upsream and downstream condition.
The numerical experiments are performed in the case of a horizontal 100 $m$ long closed uniform circular pipe with diameter 2 $m$.
The simulation starts from the inital steady state which consists of a free surface flow with a null discharge $Q_w = 0\ m^3/s$ and a water height of $y_w = 0.2\ m$.
In the following tests, the upstream water height is increasing in 25 $s$ at the height $y_w = 0.4\ m$ and the air density at the downstream boundary is taken successively as $\rho_a = 1.29349\ kg/m^3$ and $1.29349\ 10^{-2}\ kg/m^3$. The downstream discharge of water and air is kept equal $0\, m^3/s$ (as displayed on Fig. 4).
We compare then the results obtained for the present model and the Saint-Venant Equations for the water layer only, solved by the kinetic scheme at the middle of the pipe for the water heigh, the water discharge, the air pressure. The piezometric head is defined as:

$$piezo = z + h_w \quad \text{with } h_w \text{ the water height} \quad .$$



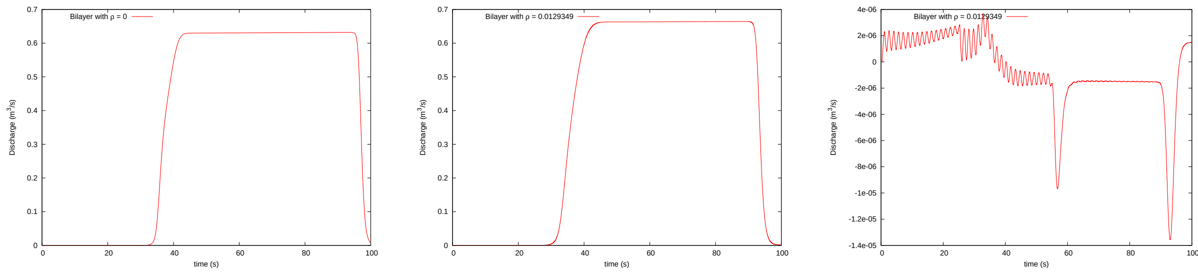

(a) Downstream Air and Water discharge (b) Upstream Water piezomeric head (c) Upstream Air discharge

Figure 4: Upstream and downstream boundary conditions



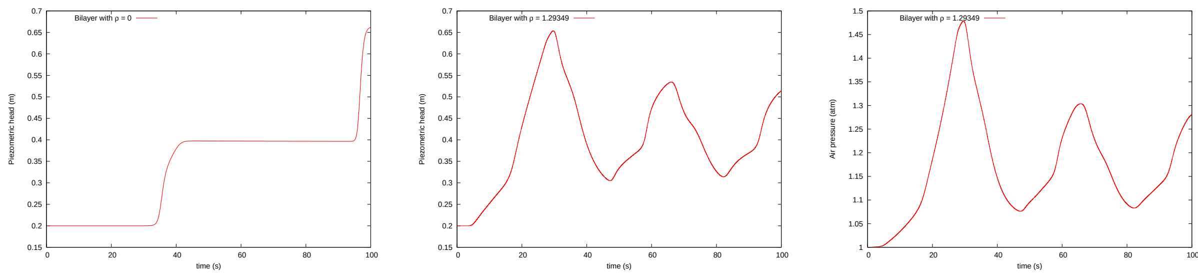

(a) Single fluid (b) Piezometric head of water (c) Air pressure

Figure 5: Piezometric head of water and air pressure at the middle of the pipe. $\rho_a = 1.29349\ 10^{-2}\ kg/m^3$



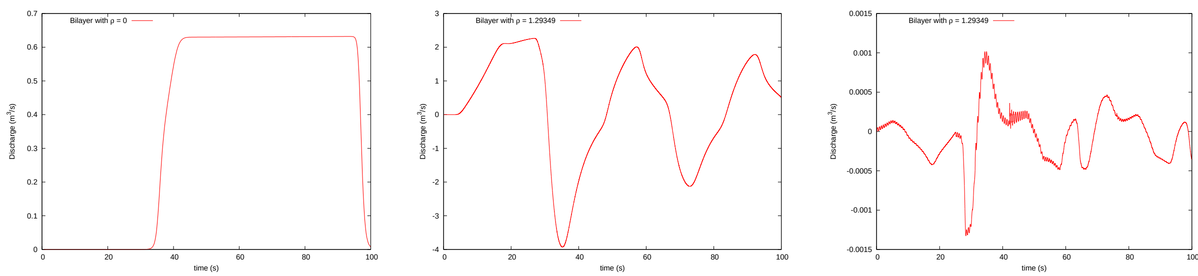

(a) Single fluid (b) Water discharge (c) Air discharge

Figure 6: Discharge of water and air at the middle of the pipe. $\rho_a = 1.29349\ 10^{-2}\ kg/m^3$




(a) Single fluid (b) Piezometric head of water (c) Air pressure

Figure 7: Piezometric head of water and air pressure at the middle of the pipe. $\rho_a = 1.29349\ kg/m^3$




(a) Single fluid (b) Water discharge (c) Air discharge

Figure 8: Discharge of water and air at the middle of the pipe. $\rho_a = 1.29349\ kg/m^3$

We can see on Fig. 5 and Fig. 6 (at the downstream and upstream boundaries $\rho_a = 1.29349\ 10^{-2}\ kg/m^3$) that the influence on the air entrainment on the water flow is negligeable whereas on Fig. 7 and Fig. 8 (at the downstream and upstream boundaries $\rho_a = 1.29349\ kg/m^3$) the air entrainment is very important : since we imposed that the discharge of air is null at the two boundaries, the air layer is pushed in the pipe, so that the air layer is compressed. On the other hand, the progression of discharge of the water is accelerated since the air layer pushes on the water layer.

# 6 Conclusion

This study is of course the first step in the comprehension and the modelisation of the air role in the transient flows in closed pipes. We have now to treat the air entrapment pocket which may be encounter in a rapid

filling process of a closed pipe: in this case a portion of the pipe will be completely filled and air pocket will be entrapped. This is the next step of our research work since in previous works, we have derived a model for pressurised flow or mixed flows in closed pipes *without taking into account the role of the air* and proposed a Roe like Finite Volume method and a kinetic scheme [4, 5, 6, 7, 8].
Next, we have to deal with the evaporation/condesation of gas or water when it is not supposed to be isothermal and lately, because in the water hammer phenomenon, large depression may occur, we have to deal with the natural cavitation problem.

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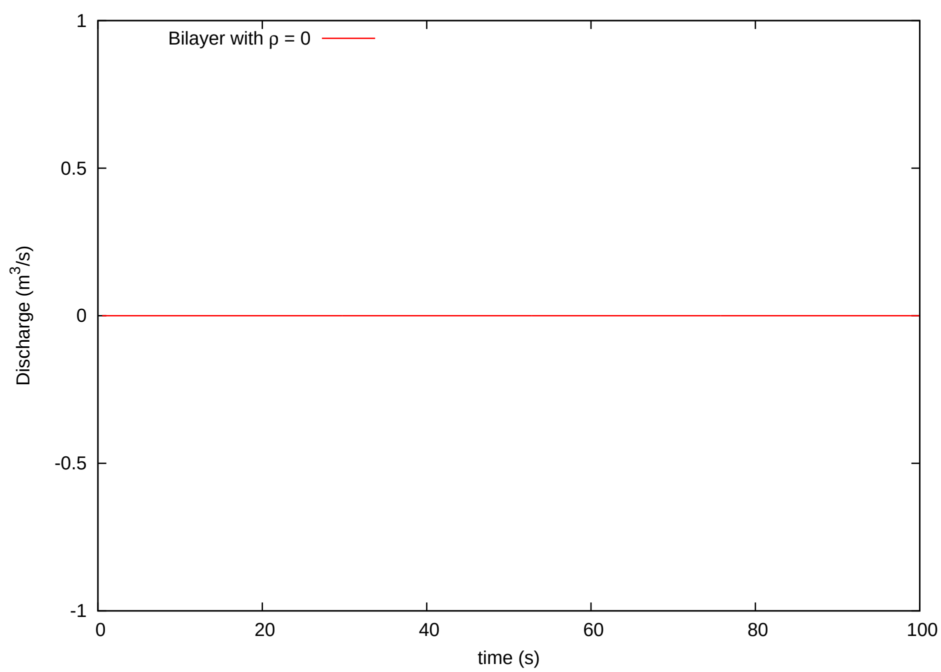
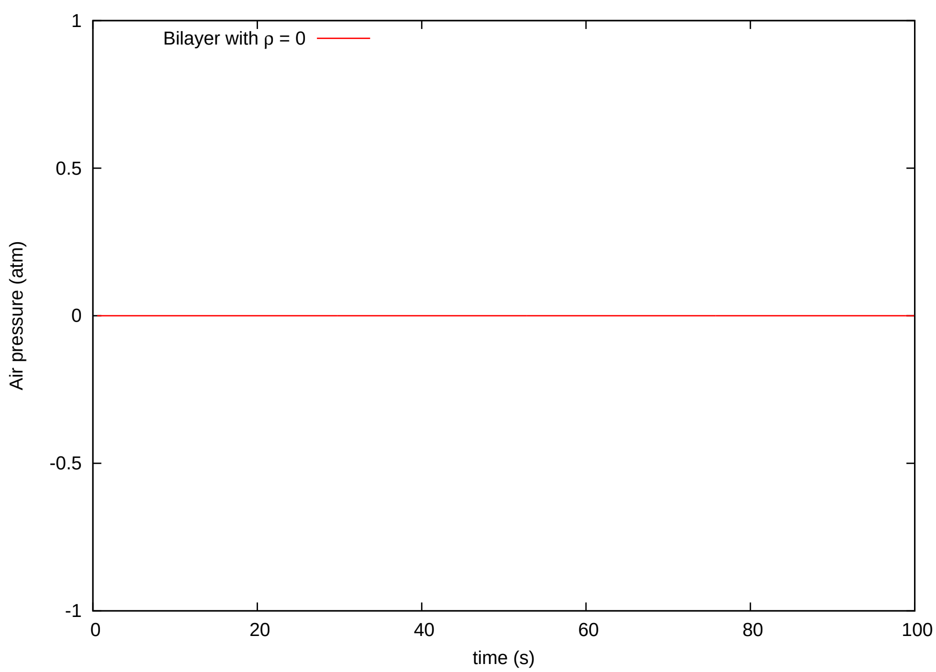

Bilayer with ρ = 0
1
0.5
0
-0.5
-1
Discharge (m3/s)
0
20
40
60
80
100
time (s)



Bilayer with ρ = 0
Air pressure (atm)
1
0.5
0
-0.5
-1
0
20
40
60
80
100
time (s)